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LOCAL SOLUTION FOR THE FLOW OF AN IDEAL FLUID NEAR A SEPARATION LINE

[LOKAL'NOE RESHENIE V OKRESTNOSTI LINII OTRYVA POTOKA IDEAL'NOI ZHIDKOSTI]

by

V. A. Malanichev

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AUTHOR'S SUMMARY

At the present time many papers have been published on numerical and analytical methods of investigating separated flows of an ideal fluid. In papers[1,2] local solutions have been given near the separation point in a two-dimensional case.

The present paper examines three-dimensional flows with a surface having tangential velocity separation. A general method is proposed for solving local problems near a separation line, and three concrete examples have been considered.

The solutions which have been obtained are of interest for the numerical calculation of the evolution of a vortex sheet.

For the first time the possibility has been indicated of breaching the '3/2' law during separation of flow from a wedge-shaped edge.

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1 STATEMENT OF THE PROBLEM

Let us consider steady three-dimensional flow having a surface of discontinuity of tangential velocity near the separation line. This line may be considered straight on scales less than the radius of curvature of the edge of the body.

Let us introduce a cylindrical system of coordinates r, ϕ, z where the axis z is directed along the separation line (Fig 1). The region of flow is broken into two regions «+» and «-» by the surface F_0. In each of them the flow potential satisfies the Laplace equation

$$\frac{\partial^2\Phi^\pm}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi^\pm}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi^\pm}{\partial\phi^2} + \frac{\partial^2\Phi^\pm}{\partial z^2} = 0 \tag{1}$$

and the following boundary conditions:

(1) non-leakage on the surface of the body $G_0^\pm(r,\phi,z) = 0$;

$$\nabla G_0^\pm \cdot \nabla\Phi^\pm = 0 \; ; \tag{2}$$

(2) non-leakage on the separation surface $F_0(r,\phi,z) = 0$:

$$\nabla F_0 \cdot \nabla\Phi^\pm = 0 \; ; \tag{3}$$

(3) the equality of static pressures on the surface $F_0(r,\phi,z) = 0$:

$$(\nabla\Phi^+)^2 = (\nabla\Phi^-)^2 \; . \tag{4}$$

In addition, we seek a solution which satisfies the Chaplygin-Zhukovskii condition of finite velocity. We seek the solution of equation (1) with boundary conditions (2) to (4) for $r \to 0$ in the form of a series:

$$\left.\begin{aligned} \Phi^+(r,\phi,z) &= \sum_i B_i(\phi,z)r^{\beta_i}, \quad \beta_i \geqslant 0 \; , \quad \beta_{i+1} > \beta_i \; , \\ \Phi^-(r,\phi,z) &= \sum_i A_i(\phi,z)r^{\alpha_i}, \quad \alpha_i \geqslant 0 \; , \quad \alpha_{i+1} > \alpha_i \; , \qquad i=0,1,\ldots \; . \end{aligned}\right\} \tag{5}$$

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The form of the surface is given as follows:

$$F_0(r,\phi,z) = \phi - \sum_i f_j(z)r^{\delta_j} , \quad \delta_{j+1} > 0 , \quad \delta_{j+1} > \delta_j , \quad j=1,2,\ldots, \tag{6}$$

where α_i, β_i, δ_j are unknown constants, while we consider that the functions A_i, B_i, f_j are infinitely differentiable with respect to ϕ, z .

2 SEPARATION FROM A SHARP EDGE OF THE PLATE

Let us examine the separation of the surface F_0 *from the sharp edge of* the plate (see Fig 1) $G_0^+ = \phi - \pi$, $G_0^- = \phi + \pi$.

Taking account of (5) equation (1) assumes the form

$$\sum_i r^{\beta_i - 2}\left(B_i\beta_i^2 + \frac{\partial^2 B_i}{\partial \phi^2}\right) = -\sum_i \frac{\partial^2 B_i}{\partial z^2} r^{\beta_i} \tag{7}$$

for the region ≪+≫ and similarly for the region ≪-≫ .

The boundary conditions (2) to (3) assume the form:

$$\frac{\partial B_i}{\partial \phi} = 0 , \quad \phi = \pi ; \quad \frac{\partial A_i}{\partial \phi} = 0 , \quad \phi = -\pi ; \tag{8}$$

$$\left(\sum_i \beta_i B_i r^{\beta_i - 1}\right)\left(\sum_i f_j \delta_j r^{\delta_j - 1}\right) = \sum_i \frac{\partial B_i}{\partial \phi} r^{\beta_i - 2}$$

$$- \left(\sum_j f_j' r^{\delta_j}\right)\left(\sum_i \frac{\partial B_i}{\partial z} r^{\beta_i}\right) , \quad \phi = \sum_i f_j r^{\delta_j} \tag{9}$$

and similarly for $\phi^-(r,\phi,z)$.

We assume that

$$\left.\frac{\partial \phi^\pm}{\partial r}\right|_{r\to 0} = \xi^\pm(z,\phi) \neq 0 .$$

Then

$$\phi^+ = B_0(\phi,z) + B_1(\phi,z)r + B_2(\phi,z)r^{\beta_2} + \ldots ,$$

$$\phi^- = A_0(\phi,z) + A_1(\phi,z)r + A_2(\gamma,z)r^{\alpha_2} + \ldots ,$$

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substituting these expansions in (7)

$$r^{-2}\frac{\partial^2 B_0}{\partial\phi^2} + r^{-1}\left[\frac{\partial^2 B_1}{\partial\phi^2} + B_1\right] + r^{\beta_2 - 2}\left[\frac{\partial^2 B_2}{\partial\phi^2} + \beta_2^2 B_2\right] = 0 \ .$$

From conditions (8) to (9) we find

$$\Phi^+(r,\phi,z) = P_0(z) + rP_1(z)\cos\phi + r^{\beta_2}(P_{21}(z)\cos\beta_2\phi + P_{22}(z)\sin\beta_2\phi) + \ldots, \qquad (10)$$

where $P_{21}(z)\sin\beta_2\pi = P_{22}(z)\cos\beta_2\pi$

$\beta_2 = 1 + \delta_1$,

$P_{22} = f_1 P_1$; $f_1(z)$, $P_i(z)$ are unknown functions.

Similarly for Φ^- we find

$$\Phi^-(r,\phi,z) = Q_0(z) + Q_1(z)\cos\phi\, r + r^{\alpha_2}(Q_{21}(z)\cos\alpha_2\phi + Q_{22}(z)\sin\alpha_2\phi) + \ldots \ , \qquad \ldots\ldots(11)$$

where $Q_{21}\sin\alpha_2\pi = -Q_{22}\cos\alpha_2\pi$,

$\alpha_2 = 1 + \delta_1$,

$Q_{22} = f_1 Q_1$ for unknown $Q_i(z)$, $f_1(z)$.

In order to determine δ_1 we substitute in the condition (4) the expansions (10) and (11) and collect the terms for identical powers of r :

$$\left[P_1^2 + P_0'^2 - Q_1^2 - Q_0'^2\right] + r^{\delta_1} 2(\delta_1 + 1)\left[P_1 P_{21} - Q_1 Q_{21}\right] + \ldots = 0 \ . \qquad (12)$$

Whence it follows $P_1 P_{21} = Q_1 Q_{21}$, since $f_1 \neq 0$, $P_1 \neq 0$, $Q_1 \neq 0$, then from (10) to (11) it follows $P_{22} \neq 0$, $Q_{22} \neq 0$, $\sin(1 + \delta_1)\pi \neq 0$, then $P_{21} = P_{22}\,\mathrm{ctg}\,(1 + \delta_1)\pi$, $Q_{21} = -Q_{22}\,\mathrm{ctg}\,(1 + \delta_1)\pi$. With regard to (12) $(P_{22}^2 + Q_{22}^2)\,\mathrm{ctg}\,(1 + \delta_1)\pi = 0$, whence $\delta_1 = (2n + 1)/2$, $n = 0,1,\ldots,$ $P_{21} = Q_{21} = 0$.

Therefore, for $r \to 0$ the surface F_0 has the following form: $\phi = f_1(z)r^{\delta_1} + \ldots,\ \delta_1 = \frac{1}{2}, \frac{3}{2}, \ldots$. The functions f_1, P_0, P_1, Q_0, Q_1 are not determined from local examination, their form is found from the full solution of the problem of flow about the body.

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Let us consider how the pressure gradient behaves, that is its component $\partial p/\partial r$. From the momentum equation we obtain:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^3}\left(\frac{\partial \Phi}{\partial \phi}\right)^2 - \frac{1}{r^2}\frac{\partial \Phi}{\partial \phi}\frac{\partial^2 \Phi}{\partial \phi \partial r} + \frac{\partial^2 \Phi}{\partial z \partial r}\frac{\partial \Phi}{\partial z} \quad . \tag{13}$$

For definiteness let $f_1 > 0$. Substituting in (13) the expression $\phi^+(r,\phi,z)$ we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \delta_1(1 + \delta_1)f_1P_1^2 \sin(1 + \delta_1)\pi\, r^{\delta_1 - 1} + \ldots .$$

For $\delta_1 = \frac{1}{2}$ and $P_1 > 0$ we have an unfavourable infinite pressure gradient on the separation line.

3 SEPARATION FROM A SMOOTH SURFACE

Now let the surface touch the smooth surface of the body along a straight line (Fig 2), then in polar coordinates

$$G_0^+ = \phi - \pi - \sum_n g_n(z)r^n \quad ,$$

$$G_0^- = \phi + \sum_n g_n(z)(-1)^{n+1}r^n \quad , \qquad n=1,2,\ldots .$$

Compared with the previous point there is a variation only in condition (2) which in the given case for the region $\langle + \rangle$ assumes the form

$$\left(\sum_n g_n n r^{n-1}\right)\left(\sum_i \beta_i B_i r^{\beta_i - 1}\right) = \sum_i \frac{\partial B_i}{\partial \phi} r^{\beta_i - 2} - \left(\sum_n g_n' r^n\right)\left(\sum_i \frac{\partial B_i}{\partial z} r^{\beta_i}\right) ,$$

$$\phi = \pi + \sum_n g_n r^n \quad ,$$

while for the region $\langle - \rangle$

$$-\left(\sum_n g_n n(-1)^{n+1} r^{n-1}\right)\left(\sum_i \alpha_i A_i r^{\alpha_i - 1}\right) = \sum_i \frac{\partial A_i}{\partial \phi} r^{\alpha_i - 2}$$

$$+\left(\sum_n g_n'(-1)^{n+1} r^n\right)\left(\sum_i \frac{\partial A_i}{\partial z} r^{\alpha_i}\right) ,$$

$$\phi = -\sum_n g_n(-1)^{n+1} r^n .$$

Repeating the reasoning of the previous point, we find the form of the surface F_0 for $r \to 0$

$$\phi = f_1(z) r^{\delta_1} + \ldots , \quad \delta_1 = \frac{1}{2}, \frac{3}{2}, \ldots .$$

The potential $\Phi^{\pm}$ is represented in the form

$$\Phi^+(r,\phi,z) = P_0(z) + rP_1(z) \cos\phi + r^{\delta_i + 1} P_2(z) \sin(1 + \delta_1)\phi + \ldots$$

$$\Phi^-(r,\phi,z) = Q_0(z) + Q_1(z) r^2 + \ldots ,$$

where $P_2 = f_1 P_1$, while Q_i, P_i, $i = 0,1,\ldots$ are unknown functions.

For $\delta_1 = \frac{1}{2}$ and $P_1 > 0$ there is an unfavourable infinite pressure gradient on the separation line.

4 SEPARATION FROM A WEDGE-SHAPED EDGE

Before examining the problem of the separation of the surface F_0 from a wedge-shaped edge, let us examine the two-dimensional non-steady problem of the form of the line of tangential velocity discontinuity $F_0(r,\phi,t) = 0$ coming away from the tip of a wedge, for $r \to 0$ (Fig 3).

Flow potentials in the regions $\langle + \rangle$ and $\langle - \rangle$ satisfy the Laplace equation

$$\frac{\partial^2 \Phi^{\pm}}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi^{\pm}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi^{\pm}}{\partial \phi^2} = 0 \tag{14}$$

and the following boundary conditions:

(1) non-leakage on the surface of the wedge

$$\frac{\partial \Phi^+}{\partial \phi} = 0 , \quad \phi = \pi - \gamma ; \quad \frac{\partial \Phi^-}{\partial \phi} = 0 , \quad \phi = -\pi ; \tag{15}$$

(2) non-leakage on the discontinuity line $F_0(r,\phi,t) = 0$,

$$\frac{\partial F_0}{\partial t} + \nabla F_0 \nabla \Phi^{\pm} = 0 \quad ; \tag{16}$$

(3) the equality of static pressures on the discontinuity $F_0(r,\phi,t) = 0$

$$\left[\frac{\partial \Phi}{\partial t}\right] + \tfrac{1}{2}[(\nabla\Phi)^2] = 0 \quad , \tag{17}$$

where $[\alpha] = \alpha^+ - \alpha^-$.

We seek for $r \to 0$ the solution to equation (14) having the boundary conditions (15) to (17) in the form of the series:

$$\left.\begin{aligned} \Phi^+ &= \sum_i B_i(\phi,t)r^{\beta_i} \ , \quad \beta_{i+1} > \beta_i \ , \quad \beta_i \geqslant 0 \ , \\ \Phi^- &= \sum_i A_i(\phi,t)r^{\alpha_i} \ , \quad \alpha_{i+1} > \alpha_i \ , \quad \alpha_i \geqslant 0 \ , \quad i=0,1,\ldots, \end{aligned}\right\} \tag{18}$$

while the form of the discontinuity line is given as

$$F_0(r,\phi,t) = \phi - \sum_i f_j(t)r^{\delta_j} \ , \quad \delta j > 0 \ , \quad \delta_{j+1} > \delta_j = 1,2,\ldots, \tag{19}$$

where δ_j, α_i, β_i are unknown constants, and the functions A_i, B_i are infinitely differentiable with respect to ϕ .

Substituting (18) in (14) we obtain:

$$\beta_i^2 B_i + \frac{\partial^2 B_i}{\partial \phi^2} = 0 \ , \qquad i=0,1,\ldots,$$

whence it follows that

$$B_i(\phi,t) = P_{i1}(t)\cos\beta_i\phi + P_{i2}(t)\sin\beta_i\phi \ . \tag{20}$$

Similarly, for ϕ^- we find

$$A_i(\phi,t) = Q_{i1}(t)\cos\alpha_i\phi + Q_{i2}(t)\sin\alpha_i\phi \ . \tag{21}$$

We suppose

$$\left.\frac{\partial\phi^-}{\partial r}\right|_{r\to 0} = \xi(t,\phi) \neq 0 \ ,$$

then

$$\phi^- = A_0(\phi,t) + A_1(\phi,t)r + A_2(\phi,t)r^{\alpha_2} + \dots \ .$$

Condition (15) taking account of (20) and (21) gives

$$\left.\begin{aligned} - P_{i1}(t) \sin \beta_i(\pi - \gamma) + P_{i2}(t) \cos \beta_i(\pi - \gamma) &= 0 \ , \\ Q_{i1}(t) \sin \alpha_i\pi + Q_{i2}(t) \cos \alpha_i\pi &= 0 \ . \end{aligned}\right\} \qquad (22)$$

Substituting (20) in condition (16), we obtain $B_0(\phi,t) = P_0(t), P_{12}(t) \equiv 0$, whence, using (22) we find

$$\beta_1 = \frac{\pi n}{\pi - \gamma} \ , \qquad n = 1,2,..,$$

and

$$\phi^+(r,\phi,t) = P_0(t) + P_1(t) \cos \beta_1\phi \, r^{\beta_i} + \dots \ , \qquad (23)$$

where $P_i(t)$ are arbitrary functions.

For ϕ^- we find in the same way

$$\phi^-(r,\phi,t) = Q_0(t) + Q_1(t) \cos\phi \, r + (Q_{21}(t) \sin\alpha_2\phi + Q_{22}(t) \cos\alpha_2\phi)r^{\alpha_2} + \dots \ , \qquad (24)$$

where $\alpha_2 = 1 + \delta_1$, $Q_1 f_1 = Q_{22}$ for arbitrary functions Q_1, f_1.

In order to determine δ_1 we use the condition (17). Substituting in it the expansions (23) and (24) and collecting terms with identical powers of r

$$\left(2P_0' - 2Q_0' - Q_1^2\right) + P_1^2\beta_1^2 r^{2(\beta_i - 1)} - 2Q_1Q_{21}\alpha_2 r^{\alpha_2 - 1} + \dots = 0 \ , \qquad (25)$$

where $Q_{21} = - Q_{22} \operatorname{ctg} \alpha_2\pi = - Q_1 f_1 \operatorname{ctg} \alpha_2\pi$, which follows from (22).

We re-write (25) in more convenient form

$$\left(2P_0' - 2Q_0' - Q_1^2\right) + P_1^2\beta_1^2 r^{2(\pi(n-1)+\gamma)/(\pi-\gamma)} + 2Q_1^2 f_1 \operatorname{ctg}\alpha_2\pi r^{\delta_1} + \dots = 0 \ . \qquad (26)$$

The value of n determines the structure of the flow in the region $\langle + \rangle$.

If $n > 1$, then $\operatorname{ctg} \alpha_2\pi = 0$, *ie* $\delta_1 = \frac{1}{2}$. If $n = 1$, which corresponds to one streamline passing through the point $x = y = 0$, then two cases are possible:

(a) $\delta_1 = 2(\beta_1 - 1)$ for $0 < \gamma < \frac{\pi}{5}$,

(b) $\delta_1 = \frac{1}{2}$ for $\gamma \geqslant \frac{\pi}{5}$.

Detailed analysis gives the following form for F_0 when $r \to 0$:

$$F_0 = \phi - \sum_m \sum_k f_{mk}(t) r^{2(\beta_1-1)m+\frac{1}{2}k}, \tag{27}$$

where $m + k = 1,2,\ldots$, $f_{01} \equiv 0$. (The author probably intends 'm and k', Ed.)

We note that for $\gamma < (\pi/5)$ and $n = 1$ from (26) we find $f_1 = -P_1^2\beta_1^2 \operatorname{ctg}(2\beta_1 - 1)/2Q_1^2 < 0$, which also leads to an unfavourable infinite pressure gradient on the lower surface of the wedge in the case of $Q_2 > 0$ at the point $x = y = 0$.

Paper[3] is devoted to the particular case examined at this point of the problem, *ie* an examination is made of the flow about a slender wing having a rhombic section with separation of the flow from the leading edge. Taking the flow as conical, the authors[3] used an unsteady analogy method and obtained unsteady self-similar flow with a parameter of self-similarity equal to unity in the (x,y) plane (see Fig 3). Owing to the complexity of the solution and the consequent awkwardness of the calculations, the authors were unable to interpret their results physically and therefore came to incorrect conclusions. They did not detect the dependence of the form of the sheet on the angle γ. The first term of the expansion (27), which in the general case was also determined incorrectly, was used in paper[4] in the numerical calculation of the self-similar separation flow about a wedge with approximation of the initial section F_0 of the sheet. This calls in question the correctness of these calculations for small values of γ.

The three-dimensional steady problem (see Fig 1) of the form of the surface of velocity discontinuity $F_0(r,\phi,z) = 0$ is solved according to the preceding method. We indicate merely the final result.

We consider that $\gamma \equiv \text{const}$. For $n > 1$, $r \to 0$: $\phi = f_1(z)r^{\frac{1}{2}} + \ldots$.

For the potentials $\phi^{\pm}$ we obtain

$$\phi^{+}(r,\phi,z) = \sum_{j=0}^{n} P_j(z)r^{2j} + P_{n+1}(z) \cos \beta_1\phi\, r^{\beta_i} + \dots ,$$

where $\beta_1 = [\pi/2(\pi - \gamma)]$, $P_j(z)(j < n)$ are expressed by δ_i, f_i, f_i' and the derivatives $P_0^{(k)}(z)$,

$$\phi^{-}(r,\phi,z) = Q_0(z) + Q_1(z) \cos \phi\, r + Q_2(z) \sin \tfrac{3}{2}\phi\, r^{3/2} + \dots ,$$

where $Q_2 = f_1Q_1$; Q_0, Q_1 are unknown functions.

For $n = 1$, $\gamma \geq 36^{\circ}$ the preceding expressions are correct, while for $n = 1$, $\gamma < 36^{\circ}$ we have

$$\phi = f_1(z)r^{2(\beta_1-1)} + \dots , \quad f_1(z) < 0 ,$$

$$\phi^{+}(r,\phi,z) = P_0(z) + P_1(z) \cos \beta_1\phi\, r^{\beta_1} + \dots ,$$

$$\phi^{-}(r,\phi,z) = Q_0(z) + Q_1(z) \cos \phi\, r + Q_2(z) \sin(2\beta_1 - 1)\phi\, r^{2\beta_1-1} + \dots ,$$

where $\beta_1 = \pi/(\pi - \gamma)$, $Q_2 = f_1Q_1$; P_0, P_1, Q_0, Q_1 are unknown functions.

Here again, for $Q_1 > 0$ we obtain an unfavourable infinite pressure gradient on the separation line in the region $\langle - \rangle$. The problems examined above permit various generalizations. Thus, the first terms of the expansions in the last problem of separation of a flow from a wedge-shaped edge do not change their form in the case $\gamma = \gamma(z)$, if $\beta_1(z) = \pi/(\pi - \gamma(z))$ is formally substituted. This leads merely to the occurrence of terms of the type $r^{2+k}\ln^k r$, the order of which is higher since $r^k \ln^k r \to 0$ for $r \to 0$. We note that this approach may be applied without any difficulties of principle to non-steady flows and flows with different Bernoulli constants in the regions $\langle + \rangle$ $\langle - \rangle$. The same applies if F_0 is a free surface.

I wish to express my profound gratitude to S.K. Betyev for discussion of the results, and a number of valuable comments.

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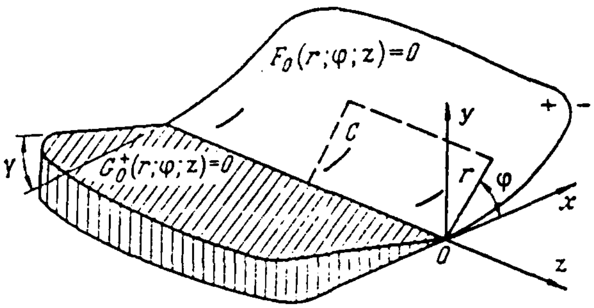

Fig 1

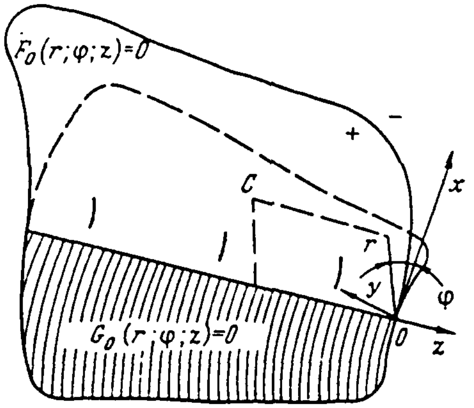

Fig 2

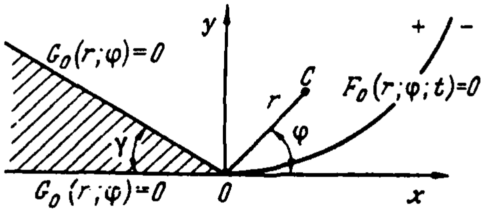

Fig 3